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A study of surfaces where two-dimensional free convection boundary layer flows occur

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Abstract — In this paper, we derive a quasilinear second-order partial differential equation which has to be satisfied by surfaces around which 3D free convection boundary layer flows can be two-dimensionalized. It is obtained from the boundary layer equations which are expressed in a system of curvilinear coordinates based upon the greatest slope lines of the gravity field and by setting to zero the source of the transverse component of the velocity. A general solution to this equation is then found by using the Lagrange–Charpit method to characterize theses surfaces. Some simple examples of explicit integration are given. © 2001 Éditions scientifiques et médicales Elsevier SAS

free convection / laminar boundary layer / curvilinear coordinates / greatest slope line / general integral / geodesic curvature

Résumé — Étude des surfaces autour desquelles la convection naturelle est bidimensionnelle. Autour de certaines surfaces particulières, un choix convenable d'un système de coordonnées curvilignes permet d'écrire les équations 3D de la couche limite de Prandlt sous forme bidimensionnelle. En convection naturelle, un tel système est basé sur les lignes de plus grande pente du champ de la pesanteur de sorte que seule la courbure géodésique de ces lignes est responsable de l'écoulement transversal. L'annulation de cette dernière permet ainsi de caractériser les surfaces autour desquelles l'écoulement est bidimensionnel. Cette opération conduit à l'établissement d'une équation aux dérivées partielles du second ordre de type hyperbolique pour définir ces surfaces. On en donne une intégrale générale au moyen de la méthode de Lagrange–Charpit. Quelques cas élémentaires d'intégration sont examinés. © 2001 Éditions scientifiques et médicales Elsevier SAS

convection naturelle / couche limite laminaire / coordonnées curvilignes / lignes de plus grande pente / intégrale générale / courbure géodésique

Nomenclature

C_1, C_2	arbitrary functions of x and y	
ds_1, ds_2	length elements along coordinate lines ξ and ζ	m
g	gravity acceleration	$m \cdot s^{-2}$
$\boldsymbol{g}$	gravity field	$m \cdot s^{-2}$
$\boldsymbol{g}_t$	tangential gravity field	$m \cdot s^{-2}$
Gr	Grashof number	
h_1, h_2	scale factors along the coordinate lines ξ and ζ	
J_1, J_2	components of $\boldsymbol{k}$ along the ξ and ζ axis	
$\boldsymbol{k}$	ascendant and vertical unit vector	
K_1, K_2	geodesic curvatures of the curves $\zeta = \text{const}$ and $\xi = \text{const}$	m^{-1}
L	characteristic length	m
$\boldsymbol{n}$	outward normal unit vector to S	
Pr	Prandtl number	
q	real constant	
S	heated surface	m^2
T	temperature	K
u, v, w	velocity components in the curvilinear coordinates system	$m \cdot s^{-1}$
x, y, z	Cartesian coordinates	m

Greek symbols

β	coefficient of thermal expansion	K^{-1}
γ	coefficient of thermal diffusivity	$m \cdot s^{-2}$
η	normal coordinate	m
μ	dynamic viscosity	$kg \cdot m^{-1} \cdot s^{-1}$
θ	normalized temperature	
ρ	density	$kg \cdot m^{-3}$
ξ, ζ	orthogonal curvilinear coordinates on S	m

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Subscripts

∞ refers to atmosphere
w refers to the surface
x, y partial derivation with respect to x, y

1. INTRODUCTION

The laminar and stationary free convection boundary layer flow around heated or cooled bodies is generally three-dimensional; this character which is purely geometric appears when writing the governing equations in an orthogonal and curvilinear coordinates system. The question we consider here is how to characterize surfaces around which the cross flow can be made to disappear and also for what coordinates system this will occur.

In forced convection, the problem of reducing the dimension of a boundary layer is discussed by many authors in the case of very small cross flows; for the subject see, for example, Chan [1], Fannelop [2], Eichelbrenner [3], Kang et al. [4], Crabbe [5], Pinkus and Cousin [6], Tsen and Arnaudon [7]. Iterative procedures are then employed to isolate and linearize the transverse equation of motion.

In free convection, an advantageous system has been suggested by Peube and Blay [8]; it is choosen so as to be tied to the steepest ascent lines of the gravity field on the surface. Because of this choice, the transversal component of the buoyancy force vanishes and the lateral component of the flow is then due only to curvature effects of the aforementioned lines. By using the hypothesis of a flow prevailing according to these lines, Peube and Blay [8] reduced the boundary layer equations to those of a two-dimensional problem. Their results are confirmed by comparison with numerical solutions and by a visualization of the wall streamlines around an isothermal and inclined circular cylinder.

In a rigorous manner and when expressed in an orthogonal coordinates system based upon the gravity field lines, the cross flow vanishes only under certain geometrical condition which consists of the annulation of the transverse variations of the longitudinal scale factor. This additional condition defines special families of surfaces among which the most obvious are cylindrical surfaces of horizontal or vertical axis and revolution surfaces of vertical axis. In the following, we are just concerned by the characterization of these surfaces.

2. GOVERNING EQUATIONS

Let us choose an arbitrary triply orthogonal coordinates system where a point M is assigned coordinates (ξ, ζ, η) such that (ξ, ζ) are the coordinates of the orthogonal projection of M on the surface S; the coordinate η measures distance from S in the direction of the outward normal to it. If h_1 and h_2 are the scale factors on S, the length elements are denoted by $\mathrm{d}s_1 = h_1\,\mathrm{d}\xi$, $\mathrm{d}s_2 = h_2\,\mathrm{d}\zeta$ and $\mathrm{d}\eta$.

In the framework of the conventional Boussinesq approximations, the dimensionless form of the free convection boundary layer equations are as follows:

$$\frac{\partial u}{\partial s_1} + \frac{\partial v}{\partial s_2} + \frac{\partial w}{\partial \eta} - K_1 u - K_2 v = 0 \tag{1}$$

$$u\frac{\partial u}{\partial s_1} + v\frac{\partial u}{\partial s_2} + w\frac{\partial u}{\partial \eta} - K_2 uv + K_1 v^2 = \frac{\partial^2 u}{\partial \eta^2} + J_1\theta \tag{2}$$

$$u\frac{\partial v}{\partial s_1} + v\frac{\partial v}{\partial s_2} + w\frac{\partial v}{\partial \eta} - K_1 uv + K_2 u^2 = \frac{\partial^2 v}{\partial \eta^2} + J_2\theta \tag{3}$$

$$Pr\left(u\frac{\partial \theta}{\partial s_1} + v\frac{\partial \theta}{\partial s_2} + w\frac{\partial \theta}{\partial \eta}\right) = \frac{\partial^2 \theta}{\partial \eta^2} \tag{4}$$

Here

$$K_1 = -\frac{1}{h_1 h_2}\frac{\partial h_2}{\partial \xi}, \qquad K_2 = -\frac{1}{h_1 h_2}\frac{\partial h_1}{\partial \zeta} \tag{5}$$

are respectively the geodesic curvatures of the curves $\xi = \text{const}$ and $\zeta = \text{const}$; u, v, w are the velocity components in the ξ, ζ, η directions, respectively, $\theta = (T - T_\infty)/(T_\mathrm{w} - T_\infty)$ is a normalized temperature; J_1 and J_2 are the components on the surface of the ascendant and vertical unit vector $\boldsymbol{k}$. For a uniformly heated surface, the following boundary conditions must be observed:

$$\begin{aligned} (u, v, w, \theta) &= (0, 0, 0, 1) \quad \text{for } \eta = 0 \\ (u, v, w, \theta) &\to (0, 0, 0, 0) \quad \text{as } \eta \to \infty \end{aligned} \tag{6}$$

In the above equations, the characteristic lengths L of the heated body and $L\,Gr^{-1/4}$ are respectively used to scale distances on S and normally to S. Gr stands for the Grashof number $(g\beta(T_\mathrm{w} - T_\infty)L^3\rho^2/\mu^2)$ where β is the coefficient of thermal expansion, ρ is the density and μ is the dynamic viscosity. Pr denotes the Prandtl number $(\mu/\rho\gamma)$ where γ is the coefficient of thermal diffusivity. The reference quantities for the velocity field are $\mu\, Gr^{1/2}/(L\rho)$ for the tangential components (u, v) and $\mu\, Gr^{1/4}/(L\rho)$ for the normal component w.

3. CHOICE OF COORDINATES AND CONDITIONS OF TWO-DIMENSIONALITY

From equation (3), we observe that the combined effects of centrifugal forces $(-K_2u^2)$ and buoyancy forces $(J_2\theta)$ generate a transverse component (v) of the velocity field. The necessary conditions for the later to vanish are

$$J_2 = 0 \quad \text{and} \quad K_2 = 0 \tag{7}$$

Under these conditions, a two-dimensional flow can exist. However, these conditions are not sufficient because a secondary flow can occur as a result of the instability of the two-dimensional flow. To make explicit these necessary conditions, let us describe the surface S by a locally nonparametric representation:

$$z + f(x, y) = 0 \tag{8}$$

where f is a twice continuously differentiable function and x, y, z are Cartesian coordinates (the z axis is vertical). Cylinders of vertical axis around which free convection is two-dimensional are not considered herein.

The condition $J_2 = 0$ is satisfied if the coordinate ξ is measured along the steepest ascent lines of the uniform gravity field $\boldsymbol{g} = -g\boldsymbol{k}$ on the surface S (*figure 1* shows the geometry of the problem). Let $\boldsymbol{n}$ be the outward unit vector normal to S, the expression of the tangential gravity field is then

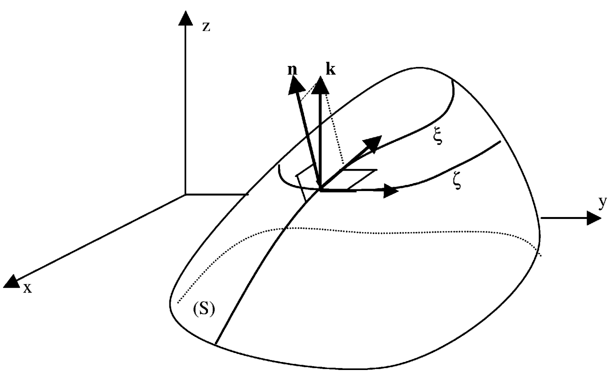

Figure 1. The geometric configuration.

$$\boldsymbol{g}_t = \boldsymbol{g} - (\boldsymbol{g}\cdot\boldsymbol{n})\boldsymbol{n} \tag{9}$$

In the system of Cartesian coordinates (x, y, z), the components of $\boldsymbol{g}_t$ are

$$\boldsymbol{g}_t \equiv \frac{g}{|\nabla f|^2 + 1}\left(f_x, f_y, -|\nabla f|^2\right) \tag{10}$$

It comes that the differential equations of the coordinate lines ξ are

$$\frac{dx}{f_x} = \frac{dy}{f_y} = -\frac{dz}{|\nabla f|^2} \tag{11}$$

The second equation expresses the appurtenance to S of the coordinate lines ξ. The first one allows us to take ζ so that $d\zeta = f_y\, dx - f_x\, dy$ since ζ must remain constant along the coordinate line ξ. On the other hand, it can be seen from the Cartesian components

$$\frac{g}{(|\nabla f|^2 + 1)^{1/2}}(-f_y, f_x, 0)$$

of the tangential vector $\boldsymbol{g}_t \wedge \boldsymbol{n}$ to the coordinate line ζ that the differential equation of this line is $dz = 0$. This implies that $d(\xi - h(z)) = 0$ for all arbitrary differentiable functions h. Therefore, we can take $\xi = z$ without loss of generality. This choice does not mean that the line z coincides with the vertical axis because it is associated with the curvilinear coordinates ζ and η and, therefore, belongs to the surface S. This allows us to express the length factors h_1 and h_2 which are given, as is well known, by the general formulae

$$h_1^2 = \left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 + \left(\frac{\partial z}{\partial \xi}\right)^2 \tag{12}$$

$$h_2^2 = \left(\frac{\partial x}{\partial \zeta}\right)^2 + \left(\frac{\partial y}{\partial \zeta}\right)^2 + \left(\frac{\partial z}{\partial \zeta}\right)^2 \tag{13}$$

Since $\xi = z$, the expression (12) takes the form

$$h_1^2 = \left(\frac{dx}{dz}\right)^2 + \left(\frac{dy}{dz}\right)^2 + 1 \tag{14}$$

The derivation with respect to z is carried out along the surface S and by taking $d\zeta = 0$. Consequently, from (8) and (11) we get

$$\frac{dx}{dz} = -\frac{f_x}{|\nabla f|^2}, \qquad \frac{dy}{dz} = -\frac{f_y}{|\nabla f|^2} \tag{15}$$

Equation (14) is then reduced to

$$h_1^2 = 1 + (\nabla f)^{-2} \tag{16}$$

and similarly equation (13) immediately leads to

$$h_2^2 = h_1^2 - 1 \tag{17}$$

With these expressions, we can now calculate the geodesic curvatures of the coordinate lines on the surface S. We find after some manipulations:

$$K_1 = \pm \frac{f_x^2 f_{xx} + f_x f_y f_{xy} + f_y^2 f_{yy}}{|\nabla f|^3 (1 + |\nabla f|^2)^{1/2}} \tag{18}$$

$$K_2 = \pm \frac{f_x f_y (f_{yy} - f_{xx}) + (f_x^2 - f_y^2) f_{xy}}{|\nabla f|^3 (1 + |\nabla f|^2)} \tag{19}$$

The vanishing of K_2 leads to the following quasilinear hyperbolic second-order partial differential equation to be satisfied by the function f:

$$f_x f_y (f_{yy} - f_{xx}) + \left(f_x^2 - f_y^2\right) f_{xy} = 0 \tag{20}$$

This equation constitutes a necessary condition for the free convection boundary layer around surfaces given locally by equation (8) to be two-dimensional. The equations (1)–(4) become then:

$$\frac{\partial u}{\partial s_1} + \frac{\partial w}{\partial \eta} - K_1 u = 0 \tag{21}$$

$$u \frac{\partial u}{\partial s_1} + w \frac{\partial u}{\partial \eta} = \frac{\partial^2 u}{\partial \eta^2} + J_1 \theta \tag{22}$$

$$Pr\left(u \frac{\partial \theta}{\partial s_1} + w \frac{\partial \theta}{\partial \eta}\right) = \frac{\partial^2 \theta}{\partial \eta^2} \tag{23}$$

where

$$J_1 = \pm \frac{|\nabla f|}{(1 + |\nabla f|^2)^{1/2}} \tag{24}$$

This expression arises from the definition: $J_1 = \boldsymbol{k} \cdot \mathrm{d}\boldsymbol{l}/|\mathrm{d}\boldsymbol{l}|$, where $\mathrm{d}\boldsymbol{l}$ is an elementary displacement along the line ξ, it is given by the following Cartesian components:

$$\mathrm{d}\boldsymbol{l} \equiv \frac{\mathrm{d}x}{f_x}\left(f_x, f_y, -|\nabla f|^2\right) \tag{25}$$

Since $\mathrm{d}\boldsymbol{l} = h_1 \, \mathrm{d}z$, it comes from (15) that $J_1 = h_1^{-1}$, which leads to (24) by applying (16).

4. GENERAL SOLUTION OF EQUATION (20)

A distinguishing feature of equation (20) is that it can be immediately integrated; one obtains:

$$f_x^2 + f_y^2 = \varphi(z) \tag{26}$$

where φ is an arbitrary positive function of the vertical coordinate z. Let us recall that (20) is obtained by annulling the geodesic curvature K_2. Hence, an alternative way to get the first integral (26) consists in solving the equation

$$\frac{\partial h_1}{\partial \zeta} = 0 \tag{27}$$

the solution of which is an arbitrary function of z and, by using (16), we obtain (26).

The Charpit method [9, 10] can be used to find a complete integral of equation (26); this first leads to

$$f_y = C_1 f_x \tag{28}$$

and hence, equation (26) can be solved for f_x. We, therefore, can write

$$\mathrm{d}z = \pm \varphi^{1/2}(z)\left(1 + C_1^2\right)^{-1/2} (\mathrm{d}x + C_1 \, \mathrm{d}y) \tag{29}$$

from which the following complete integral is deduced in the form

$$F(x, y, z, C_1, C_2) \equiv \phi(z) + \left(C_1^2 + 1\right)^{-1/2} (x + C_1 y) + C_2 = 0 \tag{30}$$

C_1 and C_2 are arbitrary constants and ϕ is an arbitrary function. The complete integral (30) now serves to derive a general one by using the Lagrange procedure [9, 10]. Assuming that C_1 and C_2 are not constant but depend on x and y, the following conditions:

$$C_2(x, y) = C_2\big(C_1(x, y)\big) \tag{31}$$

$$\frac{\partial F}{\partial C_1} + \frac{\partial F}{\partial C_2} \frac{\partial C_2}{\partial C_1} = 0 \tag{32}$$

must be observed for equation (26) to be satisfied by the complete integral (30). When associated to the condition (32) which takes the form

$$y - C_1 x + \left(C_1^2 + 1\right)^{3/2} \frac{\partial C_2}{\partial C_1} = 0 \tag{33}$$

the complete integral (30) becomes a general one.

In some elementary cases, it is possible to eliminate C_1 and C_2 to express the surface integral in its intrinsic form. When C_1 and C_2 are constant, equation (33) represents cylindrical surfaces of horizontal axis. One obtains revolution surfaces of vertical axis by choosing

$$\left(C_1^2+1\right)^{3/2}\frac{\partial C_2}{\partial C_1}=\text{const} \tag{34}$$

or

$$\frac{1}{C_1}\left(C_1^2+1\right)^{3/2}\frac{\partial C_2}{\partial C_1}=\text{const} \tag{35}$$

Another interesting case is obtained by taking

$$\left(C_1^2+1\right)^{1/2}\frac{\partial C_2}{\partial C_1}=\text{const} \tag{36}$$

We get then the equation

$$\phi(z)+\frac{x+C_1 y}{\sqrt{C_1^2+1}}+q\log\left(C_1+\sqrt{1+C_1^2}\right)=0 \tag{37}$$

with

$$C_1=\frac{x\pm[x^2-4q(y+q)]^{1/2}}{2q}$$

and q a real constant.

5. CONCLUSION

The free convection boundary layer equations are first recalled in a system of curvilinear coordinates to underline the sources of the transverse flow.

By choosing the greatest slope line of the gravity field as one of the tangential coordinate lines, the flow becomes two-dimensional around a certain family of surfaces described by a second-order hyperbolic partial differential equation. This equation is found to have a complete integral which leads to a general solution. In some elementary cases the surface equation is explicited.

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